The quantum fidelity for the time-dependent singular quantum oscillator

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THE QUANTUM FIDELITY FOR THE TIME -PERIODIC SINGULAR HARMONIC OSCILLATOR

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Résumé

In this paper we perform an exact study of "Quantum Fidelity" (also called Loschmidt Echo) for the time-periodic quantum Harmonic Oscillator of Hamiltonian :

$$\hat{H}_g(t) := \frac{P^2}{2} + f(t)\frac{Q^2}{2} + \frac{g^2}{Q^2} \tag{1}$$

when compared with the quantum evolution induced by $\hat{H}_0(t)$ $(g = 0)$, in the case where f is a T-periodic function and g a real constant. The reference (initial) state is taken to be an arbitrary "generalized coherent state" in the sense of Perelomov. We show that, starting with a quadratic decrease in time in the neighborhood of $t = 0$, this quantum fidelity may recur to its initial value 1 at an infinite sequence of times t_k . We discuss the result when the classical motion induced by Hamiltonian $\hat{H}_0(t)$ is assumed to be stable versus unstable. A beautiful relationship between the quantum and the classical fidelity is also demonstrated.

1 Introduction

A growing interest has been devoted recently on the study of the so-called "Quantum Fidelity"

$$F_g(t) := \langle U_0(t,0)\psi, U_g(t,0)\psi \rangle \tag{2}$$

for some reference state ψ that we take as initial wavepacket, $U_0(t,0)$ being the quantum unitary evolution operator induced by some Hamiltonian , and $U_g(t,0)$ the quantum evolution for a perturbation of it, g being the size of the perturbation. The long-time behavior of $F_g(t)$ is of particular interest, and it has been studied for a large class of (time independent) Hamiltonians in a more or less heuristic way, and this long-time behavior has been suggested to depend sensitively on the **regular** versus **chaotic** motion of the underlying classical motion (see the bibliography).

In most of these heuristic works ([1], [3], [4], [6], [7], [11], [12], [13], [15], [16], [17], [19], [20], [21], [22], [26], [27], [28], [29], [30], [32], [33], [34], [35], [36], [38], [39], [42], [41], [44], [45]), it is claimed that the Quantum Fidelity decays very fast to zero as time grows, when the underlying classical (unperturbed) dynamics is generically chaotic.

Although the short time decay of the fidelity is rather well understood ([43]), the arguments put forward in the above cited works are not entirely convincing, since they are either purely numerical, or extrapolate the "short time" behavior to guess the (Gaussian or exponential) decay at ∞.

In other approaches ([25], [20], [42]), the case of integrable or regular systems is considered as well, and seems to indicate the occurrence of an anomalous power law decay. Moreover, in a case of nearly integrable system a **recurrence** of fidelity has been exhibited ; in this case the Quantum Fidelity manifests recurrences very close to the initial value 1 as time evolves.

Thus it is a very intriguing subject of high interest to know more about the complete time behavior of the Quantum Fidelity when the underlying dynamics is chaotic versus regular. To our knowledge, no rigorous approaches of this topics have been attempted yet. It would be highly desirable to have a mathematically explicit description of the long time behavior of the Quantum Fidelity, although it is a very difficult task.

This paper is a first attempt towards this rigorous understanding. It relies on a rather simplistic class of Hamiltonians for which the perturbed as well as unperturbed quantum dynamics can be explicitly solved in terms of the classical dynamics. Moreover the reference quantum states are taken in a suitable large class of quantum states known as "generalized Coherent States" in the sense of Perelomov. The results are rather surprising :

- for a large class of reference states, the Quantum Fidelity **never decreases to 0**, but instead remains bounded from below by some constant.
-in the unstable case, the Quantum Fidelity either decays exponentially fast to some non-zero constant, as $t \to \infty$, or manifest strong recurrences to 1.
-in the stable case the Quantum Fidelity always manifests strong recurrences to 1 as time evolves.

These facts are strongly related to the underlying $SU(1,1)$ structure underlying the corresponding Hamiltonians, as can be seen from the work of Perelomov ([24]).

We also are able to show a strong relationship between "Quantum and Classical Fidelity" for this specific situation.

To complete this Introduction, let us notice that a notion of Classical Fidelity that "mimics" the Quantum Fidelity has been proposed in the literature ([14], [2], [37]), where decay properties similar to those of the Quantum Fidelity appear. Thus it would be desirable to understand more deeply the relationships between the Classical and Quantum Fidelity on a firm mathematical basis. We shall pursue this investigation in future publications, notably in the semiclassical limit (see [10], [8]).

2 Calculus of the Quantum Fidelity

Let us consider the following operators in $\mathcal{H} = L^2(\mathbb{R})$ with suitable domains (see [24]) :

$$K_0 = \frac{Q^2 + P^2}{4} + \frac{g^2}{2Q^2} \tag{3}$$

$$K_\pm = \frac{Q^2 - P^2}{4} \mp i\frac{Q.P + P.Q}{4} - \frac{g^2}{2Q^2} \tag{4}$$

where Q is the usual multiplication operator by x and

$$P := -i\frac{d}{dx}$$

K_0 and $K_\pm$ satisfy the usual commutation rules of $SU(1,1)$ algebra, namely

$$[K_0, K_\pm] = \pm K_\pm \tag{5}$$

$$[K_-, K_+] = 2K_0 \tag{6}$$

We define the "generalized coherent states" (squeezed states) as follows : given any $\beta \in \mathbb{C}$

$$S_\beta = \exp(\beta K_+ - \bar{\beta} K_-) \tag{7}$$

$$\psi_\beta = S(\beta)\psi \tag{8}$$

ψ being a normalized state in $\mathcal{H}$ such that

$$K_-\psi = 0 \tag{9}$$

$$2K_0\psi = (\alpha + \frac{1}{2})\psi \tag{10}$$

with

$$\alpha := \frac{1}{2} + \sqrt{\frac{1}{4} + 2g^2} \tag{11}$$

We shall focus on the following cases $g = 1,\ g = \sqrt{3}$, where ψ has the following form :

$$\varphi(x) = c_1 x^2 e^{-x^2/2} \tag{12}$$

$$\chi(x) = c_2 x^3 e^{-x^2/2} \tag{13}$$

with

$$c_1 = \sqrt{\frac{4}{3}} \tag{14}$$

$$c_2 = \sqrt{\frac{8}{15}} \tag{15}$$

(omitting the factors in π).

This makes φ and χ to be (normalized to 1) finite linear combinations of Hermite functions.

It has been shown (see [24]) that for $\psi = \varphi,\ \psi_\beta$ has the following form :

$$\psi_\beta = c_1 x^2 e^{-2(u-\epsilon)} \exp\left(-\frac{5i\theta}{2} + i\frac{\dot{u}x^2}{2} - \frac{1}{2}(u-\epsilon) - \frac{1}{2}(xe^{-(u-\epsilon)})^2\right) \tag{16}$$

where the constants $u, \dot{u}, \theta, \epsilon$ are suitably determined from $\beta \in \mathbb{C}$, whereas for $\psi = \chi$, ψ_β is :

$$\psi_\beta = c_2 x^3 e^{-3(u-\epsilon)} \exp\left(-\frac{7i\theta}{2} + i\frac{\dot{u}x^2}{2} - \frac{1}{2}(u-\epsilon) - \frac{1}{2}(xe^{-(u-\epsilon)})^2\right) \tag{17}$$

Now the evolutions of ψ_β with respect to $U_1(t,0)$ and $U_{\sqrt{3}}(t,0)$ respectively, together with $U_0(t,0)$ are completely explicit.

Consider a complex solution of the classical equations of motion induced by Hamiltonian $\hat{H}_0(t)$:

$$\ddot{x} + fx = 0 \tag{18}$$

and look for it in the form

$$x = e^{u+i\theta} \tag{19}$$

where the functions $t \mapsto u$ and $t \mapsto \theta$ are **real**.

We assume for $u(t)$ and $\theta(t)$ the following initial data :

$$u(0) = u_0$$

$$\dot{u}(0) = \dot{u}_0$$

$$\theta(0) = \theta_0$$

$$\dot{\theta}(0) = e^{-2(u_0-\epsilon)}$$

Since f is real, the wronskian of x anf $\bar{x}$ is constant and equals

$$2ie^{2\epsilon} = 2i\dot{\theta}(t)e^{2u(t)} \tag{20}$$

This yields

$$\ddot{x} = [\ddot{u} + i\ddot{\theta} + (\dot{u} + i\dot{\theta})^2]x = [\ddot{u} + \dot{u}^2 - e^{-4(u-\epsilon)}]x = -fx \tag{21}$$

and therefore u obeys the following differential equation

$$\ddot{u} + \dot{u}^2 - e^{-4(u-\epsilon)} + f = 0 \tag{22}$$

We have the following result :

Lemma 2.1 *Let $\hat{H}_g = \frac{P^2+Q^2}{2} + \frac{g^2}{Q^2}$. Then the quantum propagator for $\hat{H}_g(t)$ is of the following form :*

$$U_g(t,0) = e^{i\dot{u}Q^2/2}e^{-i(u-\epsilon)(Q.P+P.Q)/2}e^{-i(\theta-\theta_0)\hat{H}_g}e^{i(u_0-\epsilon)(Q.P+P.Q)/2}e^{-i\dot{u}_0Q^2/2} \tag{23}$$

The same formula holds for the propagator $U_0(t,0)$ of $\hat{H}_0(t)$ with $\hat{H}_g$ replaced by $\hat{H}_0$.

Proof : Let us denote

$$V(t) := e^{i\dot{u}Q^2/2}e^{-i(u-\epsilon)(Q.P+P.Q)/2}e^{-i\theta\hat{H}_g} \tag{24}$$

We have :

$$i\frac{d}{dt}V(t) = \left(-\ddot{u}\frac{Q^2}{2} + \frac{\dot{u}}{2}e^{i\dot{u}Q^2/2}(Q.P + P.Q)e^{-i\dot{u}Q^2/2}\right)V(t) \tag{25}$$

$$+\left(\dot{\theta}e^{i\dot{u}Q^2/2}e^{i(u-\epsilon)(Q.P+P.Q)/2}\hat{H}_g e^{-i(u-\epsilon)(Q.P+P.Q)/2}e^{-i\dot{u}Q^2/2}\right)V(t)$$

and since

$$e^{i\dot{u}Q^2/2}Pe^{-i\dot{u}Q^2/2} = P - \dot{u}Q \tag{26}$$

we have :

$$e^{i\dot{u}Q^2/2}(P.Q + Q.P)e^{-i\dot{u}Q^2/2} = P.Q + Q.P - 2i\dot{u}Q^2 \tag{27}$$

Therefore the first line in the RHS of (25) is

$$\left((-\frac{\ddot{u}}{2} - \dot{u}^2)Q^2 + \frac{\dot{u}}{2}(Q.P + P.Q)\right)V(t) \tag{28}$$

Furthermore

$$e^{-i(u-\epsilon)(Q.P+P.Q)/2}\hat{H}_g e^{-i(u-\epsilon)(P.Q+Q.P)/2} = \left(\frac{P^2}{2} + \frac{g^2}{Q^2}\right)e^{2(u-\epsilon)} + \frac{Q^2}{2}e^{-2(u-\epsilon)} \tag{29}$$

which implies that the second line in the RHS of (25) is

$$\left(\frac{1}{2}(P - \dot{u}Q)^2 + \frac{g^2}{Q^2} + \frac{Q^2}{2}e^{-4(u-\epsilon)}\right)V(t) \tag{30}$$

where we have used that

$$\dot{\theta} = e^{-2(u-\epsilon)}$$

Collecting the different terms, we get :

$$i\frac{d}{dt}V(t) = \left(\frac{P^2}{2} + \frac{g^2}{Q^2} + \frac{Q^2}{2}(-\ddot{u} - \dot{u}^2 + e^{-4(u-\epsilon)})\right)V(t) \tag{31}$$

$$= \left(\frac{P^2}{2} + \frac{g^2}{Q^2} + f(t)\frac{Q^2}{2}\right)V(t)$$

using (22). □

We shall now consider the "quantum fidelity" in two particular cases $g = 1$ and $g = \sqrt{3}$, starting respectively with the initial states $\psi_{\beta,1} = S(\beta)\varphi$, and $\psi_{\beta,2} = S(\beta)\chi$:

$$F_1(t) = \langle U_0(t,0)\psi_{\beta,1}, U_g(t,0)\psi_{\beta,1}\rangle \tag{32}$$

and similarly for $F_2(t)$ with $\psi_{\beta,1}$ replaced by $\psi_{\beta,2}$.

Theorem 2.2 *(i) Let $g = 1$ and $\psi_{\beta,1} = S(\beta)\varphi$. Then we have*

$$F_1(t) = \frac{2}{3} + \frac{1}{3}e^{-2i(\theta(t)-\theta(0))} \tag{33}$$

(ii) Let $g = \sqrt{3}$ and $\psi_{\beta,2} = S(\beta)\chi$. Then we have :

$$F_2(t) = \frac{2}{5} + \frac{3}{5}e^{-3i(\theta(t)-\theta(0))} \tag{34}$$

Proof : Since x^2 is expanded as

$$x^2 = \frac{1}{4}H_2(x) + \frac{1}{2}H_0(x)$$

it is clear that :

$$U_0(t,0)\psi_{\beta,1} = c_1 \exp\left(\frac{i}{2}\dot{u}(t)x^2 - \frac{1}{2}(u(t)-\epsilon)\right) \tag{35}$$
$$\times \left\{\frac{1}{4}e^{-5i\theta(t)/2}H_2(xe^{-(u(t)-\epsilon)}) + \frac{1}{2}e^{-i\theta(t)/2-2i\theta(0)}\right\} \exp\left(-\frac{1}{2}x^2e^{-2(u(t)-\epsilon)}\right)$$

and

$$U_g(t)\psi_{\beta,1} = V(t)\varphi \tag{36}$$
$$= c_1 \exp\left(-\frac{5}{2}i\theta(t) + \frac{i}{2}\dot{u}(t)x^2 - \frac{1}{2}(xe^{-(u(t)-\epsilon)})^2\right)$$

from which we deduce that

$$F_1(t) = c_1^2 \int \left[x^2(x^2 - \frac{1}{2}) + \frac{e^{-2i(\theta(t)-\theta(0))}}{2}x^2\right] e^{-x^2}\, dx \tag{37}$$
$$= \frac{4}{3}\left(\frac{1}{2} + \frac{1}{4}e^{-2i(\theta(t)-\theta(0))}\right) = \frac{2}{3} + \frac{1}{3}e^{-2i(\theta(t)-\theta(0))}$$

(ii) follows from a very similar calculation using that

$$H_3(x) = 8x^3 - 12x$$

□

Let us now assume that g has an arbitrary real value (not specifically of the form $g = \sqrt{k(k+1)/2}$ for $k \in \mathbb{N}$ which gives rise to integer values of $\alpha = \frac{1}{2} + \sqrt{\frac{1}{4} + 2g^2}$). Then the state $\psi := cx^\alpha e^{-x^2/2}$ is no longer a finite linear combination of Hermite functions. It has nevertheless an infinite expansion on the basis of Hermite functions φ_n :

$$\psi = \sum_{n=0}^{\infty} \lambda_n \varphi_n$$

and one can establish the following general result about the corresponding "quantum fidelity" :

Theorem 2.3

$$F_g(t) = \exp(-i\alpha(\theta(t) - \theta(0))) \sum_{n=0}^{\infty} |\lambda_n|^2 \exp(in(\theta(t) - \theta(0)))$$

which, since $\sum_{n=0}^{\infty} |\lambda_n|^2 = 1$ *, returns in absolute value to 1 as long as* $\theta(t) - \theta(0) = 0 \ (mod\ 2\pi)$*. If* $\alpha = p/q$ *is rational, then the quantum fidelity recurs exactly to 1 (not only in absolute value) provided that* $\theta(t) - \theta(0) = 0 \ (mod\ 2q\pi)$*.*

Proof : Again according to ref. [24], we have

$$\psi_\beta := S(\beta)\psi = e^{-i(\alpha+\frac{1}{2})\theta_0} D(u_0)\psi$$

where by $D(u)$ we denote the following unitary operator

$$D(u) := e^{i\dot{u}Q^2/2} e^{-i(u-\epsilon)(Q.P+P.Q)/2}$$

Then

$$U_0(t,0)\psi_\beta = D(u_t) e^{-i(\theta_t-\theta_0)\hat{H}_0} \sum_{n=0}^{\infty} \lambda_n e^{-i\theta_0(\alpha+\frac{1}{2})} \varphi_n$$

$$= D(u_t) \sum_{n=0}^{\infty} \lambda_n e^{-i\theta_t(n+\frac{1}{2})+i\theta_0(n-\alpha)} \varphi_n$$

whereas

$$U_g(t,0)\psi_\beta = e^{-i\theta_t(\alpha+\frac{1}{2})} D(u_t)\psi$$

so that :

$$\langle U_0(t,0)\psi_\beta, U_g(t,0)\psi_\beta\psi_\beta\rangle = \sum_{n,m} \bar{\lambda}_n \lambda_m e^{i\theta_t(n+\frac{1}{2})-i\theta_0(n-\alpha)-i\theta_t(\alpha+\frac{1}{2})} \langle \varphi_n, \varphi_m\rangle$$

$$= \sum_{n=0}^{\infty} |\lambda_n|^2 e^{i(\theta_t-\theta_0)(n-\alpha)}$$

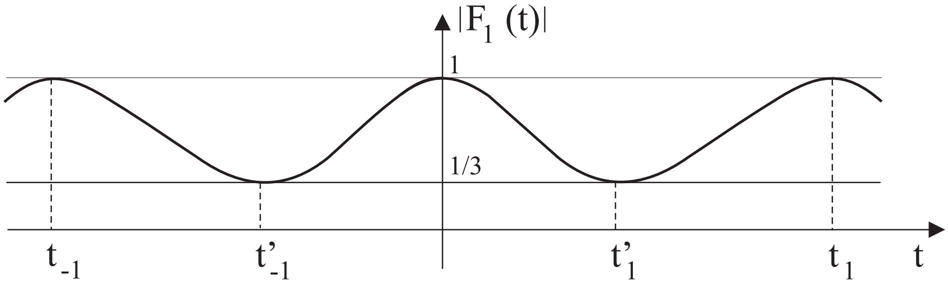
□

3 Discussion of the Result

Since f is a T-periodic function, Floquet analysis applies to equation (18) which is nothing but the well-known Hill's equation. Depending to the parameters characterizing the function f (as for example γ and δ in the case of Mathieu equation where $f(t) = \gamma + \delta \cos \omega t$), the solutions can be either **stable** or **unstable**. In all cases the quantum fidelity $F_1(t)$ and $F_2(t)$ are bounded from below by some positive constant in absolute value, and therefore never decrease to zero.
The phase $\theta(t)$ is determined by

$$\theta(t) - \theta(0) = e^{2\epsilon} \int_0^t e^{-2u(s)} \, ds \tag{38}$$

In the **stable case**, the function $t \mapsto u(t)$ is T-periodic. Therefore $\theta(t)$ grows from $-\infty$ to $+\infty$ as time evolves from $-\infty$ to $+\infty$. It follows that there exists an infinite sequence of times $\{t_k\}$ where $F_1(t)$ recurs to its initial value 1, and an infinite squence $\{t'_k\}$ where $|F_1(t)|$ attains its minimum 1/3. The same statement holds for $F_2(t)$ where 1/3 is replaced by 1/5.



stable case, g=1

In the **unstable case** there is some positive Lyapunov exponent λ and some real solution of Hill's equation such that

$$x(T) = e^{\lambda T} x(0)$$

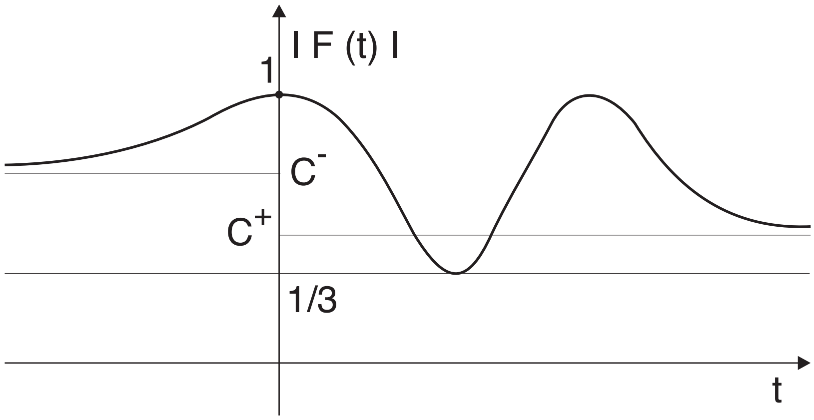
$$\dot{x}(T) = e^{\lambda T} \dot{x}(0)$$

Here, since we deal with complex solutions $x(t)$ such that the Wronskian of x and $\bar{x}$ is non zero, we deduce that $|x(t)| > 0$. Moreover, depending on the instability zone, $|x(t)|^{-2} = e^{-2u(t)}$ can be integrable near $\pm\infty$, or diverge at $\pm\infty$. These topics are clearly detailed in ref. [23]. In the first case the conclusion is that there exists two constants $\theta_\pm$ such that

$$\theta(t) \rightarrow \theta_\pm \text{ as } t \rightarrow \pm\infty \tag{39}$$

This is the case for the Inverted Harmonic Oscillator ($f = -1$) that we describe in the last Section.

Therefore the Quantum Fidelity in this case behaves generically as described in the following picture :



If the instability zone in which the solution $x(t)$ lies is such that $|x(t)|^{-2}$ is not integrable near $\pm\infty$, then the figure is typically similar to that of the stable case, which shows infinite recurrences to 1 of the Quantum Fidelity.

4 Link with the "Classical Fidelity"

We can call "classical infidelity" the discrepancy between two classical trajectories, along their evolution, when they merge from the **same initial phase-space point** at $t = 0$. The classical fidelity is thus the possible crossing of the two trajectories governed by $H_0(t)$ and $H_g(t)$, in phase-space, as time evolves.
From now on we assume that ϵ of Section 2 is determined by $(g > 0)$:

$$e^{2\epsilon} = g\sqrt{2} \tag{40}$$

Proposition 4.1 *Let $x(t)$ be a* **real** *trajectory for the time-periodic Hamiltonian $H_0(t)$, and $y(t)$ a* **real** *trajectory for Hamiltonian $H_g(t)$. Assume in addition that they satisfy $x(0) = y(0) \in \mathbb{R}$, $\quad \dot{x}(0) = \dot{y}(0) \in \mathbb{R}$. Then we have :*

$$|x(t) - y(t)| = |y(t)|(1 - \cos\tilde{\theta}(t)) \tag{41}$$

where $\tilde{\theta}(t)$ is defined as follows :

$$\tilde{\theta}(t) := g\sqrt{2}\int_0^t y(s)^{-2}\, ds \equiv \theta(t) - \theta(0) \tag{42}$$

Proof : Let $z(t)$ be a complex solution of equation

$$\ddot{z} + fz = 0 \tag{43}$$

It can be written as $z(t) = e^{u+i\tilde{\theta}}$ as above, where $t \mapsto u$ and $t \mapsto \tilde{\theta}$ are **real** functions. Assume that $x(t)$ is such that $x(0) > 0$. Then it is easy to see that $x(t) = \Re z(t) = e^{u(t)} \cos \tilde{\theta}(t)$ is a solution of equation (43), and that the positive function $y(t) := e^{u(t)}$ is a solution of

$$\ddot{y} + fy - \frac{2g^2}{y^3} = 0 \tag{44}$$

which means that it defines a classical trajectory for $H_g(t)$.

Namely we get from equ. (42) that

$$\frac{d}{dt}\tilde{\theta}(t) = g\sqrt{2}e^{-2u(t)} \tag{45}$$

and therefore

$$\ddot{x} = (\ddot{u} + \dot{u}^2 - 2g^2 e^{-4u})x = -fx \tag{46}$$

so that

$$(\ddot{u} + \dot{u}^2 - 2g^2 e^{-4u} + f)e^u = 0 \tag{47}$$

which is nothing but equ. (44) noting that

$$\ddot{y} = (\ddot{u} + \dot{u}^2)y \tag{48}$$

Moreover, they satisfy $x(0) = y(0), \qquad \dot{x}(0) = \dot{y}(0)$, and

$$|x(t) - y(t)| = y(t)(1 - \cos \tilde{\theta}(t))$$

In the case where $x(0) < 0$, we just take $x(t) = -\Re z(t)$ and $y(t) = -e^{u(t)}$, which completes the result. □

Conclusion : The classical infidelity vanishes for $\tilde{\theta}(t) = 2k\pi$ ($k \in \mathbb{Z}$), which precisely gives rise to recurrences to 1 of the "quantum fidelity" (Theorem 2.3). We expect this remarkable property to be true in more general situations, in particular in the Semiclassical Regime (see [10]).

By "vanishing of the classical infidelity" (thus "classical fidelity") we meant that given any solution of the unperturbed dynamics, there exists a solution of the perturbed one that coincides with it at the origin, and at any values of time solving the equation

$$g\sqrt{2}\int_0^t ds\ y(s)^{-2} = 0 \ (\mathrm{mod}\ 2\pi)$$

One may ask whether this holds true for **general** solution $x(t), \quad y(t)$ of equ. (43-44). This is answered in the following particular cases below.

• **Particular case** $f = 1$

(The case $f = \omega^2$ could be treated as well.)
When we apply the result above, we find that solutions $x(t) = A\cos t$ and $y = A$ coincide for $t = 2k\pi,\ k \in \mathbb{Z}$, and that y obeys equation (44) provided $A = (2g^2)^{1/4}$. One shows that a more general result holds, involving **general solutions** of the equations under consideration.

The general solution of equ. (44) is of the form (apart from the sign before the square root) :

$$y(t) = \sqrt{\alpha + \beta\cos 2t + \gamma\sin 2t} \tag{49}$$

with $\alpha,\ \beta,\ \gamma$ related to each other by the relation

$$\alpha^2 - (\beta^2 + \gamma^2) = 2g^2 \tag{50}$$

It obeys the initial data

$$y(0) = \sqrt{\alpha + \beta}$$

$$\dot{y}(0) = \frac{\gamma}{\sqrt{\alpha + \beta}}$$

and the conserved energy is simply α.

We take as real solution of equ. (43) (Harmonic Oscillator) :

$$x(t) = \sqrt{\alpha + \beta}\cos t + \frac{\gamma}{\sqrt{\alpha + \beta}}\sin t \tag{51}$$

Which has the same initial data as $y(t)$. Both functions being 2π- periodic, the generic “classical infidelity” vanishes when $t = 2k\pi,\ k \in \mathbb{Z}$.

• **Particular case** $f = -1$

(The case $f = -\omega^2$ could be trated as well)
Any complex solution of the differential equation (Inverted Harmonic Oscillator)

$$\ddot{z} - z = 0 \tag{52}$$

can be written in the form

$$z(t) = (a + ib)e^t + (c + id)e^{-t} \tag{53}$$

where $a,\ b,\ c,\ d$ are real constants. We define :

$$Z(t) := |z(t)|^2$$

One can prove that $y(t) = \sqrt{Z(t)}$ obeys the differential equation

$$\ddot{y} - y - \frac{2g^2}{y^3} = 0 \tag{54}$$

with

$$g^2 = 2(ad - bc)^2 > 0 \tag{55}$$

It is important to note that this implies $Z(t) > 0, \quad \forall t$.

Define $x(t) := y(t) \cos\theta(t)$, with $\theta(0) = 0$. Then, clearly

$$x(0) = y(0)$$

$$\dot{x}(0) = \dot{y}(0)$$

We want $x(t)$ to be a real solution of equ.(52). Since

$$\ddot{x} - x = \frac{1}{\sqrt{Z}} \left(\frac{\ddot{Z}}{2} - \frac{\dot{Z}^2}{4Z} - Z - \dot{\theta}^2 Z \right) \cos\theta - \left(\frac{\dot{\theta}\dot{Z}}{\sqrt{Z}} + \ddot{\theta}\sqrt{Z} \right) \sin\theta \tag{56}$$

and since Z obeys :

$$\frac{\ddot{Z}}{2} - \frac{\dot{Z}^2}{4Z} - Z - \frac{2g^2}{Z} = 0 \tag{57}$$

the RHS of equ. (56) vanishes provided

$$\dot{\theta}^2 = 2g^2 Z^{-2} \tag{58}$$

Then $x(t)$ and $y(t)$ are two trajectories for Hamiltonians $\frac{P^2 - Q^2}{2}$ and $\frac{P^2 - Q^2}{2} + \frac{g^2}{Q^2}$ respectively, with the same initial data, and we have

$$|x(t) - y(t)| = y(t)(1 - \cos\theta(t)) \tag{59}$$

Using the particular solution of equ. (52)

$$z(t) = \frac{1+i}{2} e^t + \frac{1-i}{2} e^{-t}$$

that can be rewritten as :

$$z(t) = \sqrt{\cosh 2t}\, \exp(\frac{i}{2} \arcsin \tanh 2t)$$

i.e. with $u(t) = \frac{1}{2} \log\cosh 2t$ (which satisfies $u(0) = \dot{u}(0) = 0$), the formula (23) can be rewritten in the simple form

$$U_g(t,0) = \exp\left(\frac{i}{2} Q^2 \tanh 2t\right) \exp\left(\frac{i}{4}(Q.P + P.Q)\log\cosh 2t\right) \exp\left(-\frac{i}{2}\hat{H}_g \arcsin\tanh 2t\right) \tag{60}$$

which holds true for $g = 0$, and $g = 1/\sqrt{2}$. For $g = 0$, this is nothing but the well-known Mehler's Formula.

Study of $\theta(t)$

Recall that the fidelity $F_g(t)$ strongly depends on the reference state via the constants $a,\ b,\ c,\ d$

$$\theta(t) = g\sqrt{2}\int_0^t \frac{ds}{(a^2+b^2)e^{2s} + (c^2+d^2)e^{-2s} + 2(ac+bd)} \tag{61}$$

$$= \left(\arctan\left(\frac{ac+bd+(a^2+b^2)e^{2t}}{|ad-bc|}\right) - \arctan\left(\frac{ac+bd+(a^2+b^2)}{|ad-bc|}\right)\right)$$

This implies that $\theta(t) \to \theta_\pm$ as $t \to \pm\infty$ exponentially fast in the future and in the past. The calculus is especially simple in the particular case where we choose $a = d = g/\sqrt{2},\ c = b = 0$:

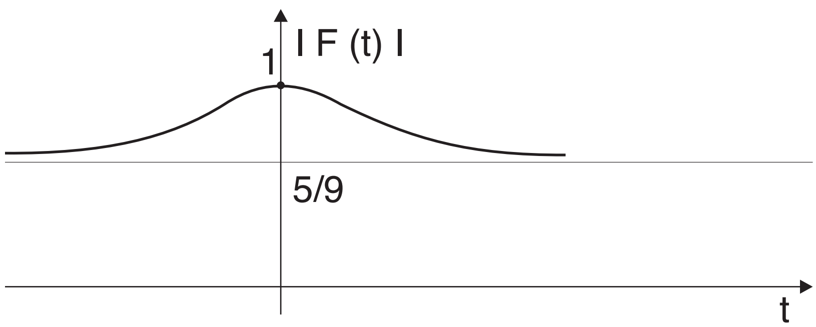
$$\theta(t) = \arctan(e^{2t}) - \frac{\pi}{4} \tag{62}$$

No classical fidelity happens, and the square of the quantum fidelity in the case $g = 1$ is nothing but

$$|F_1(t)|^2 = \frac{5}{9} + \frac{8}{9(e^{2t}+e^{-2t})} \sim \frac{5}{9} + \frac{4}{9}e^{-2|t|} \tag{63}$$

as $t \to \pm\infty$.

Here the symmetry is perfect between the future and the past, and the Quantum Fidelity has no recurrences to 1, but instead decays exponentially fast to 5/9.



5 CONCLUSION

It has been suggested in many recent works that the type of **decay** of the Quantum Fidelity may help to **discriminate** between chaotic or regular underlying classical motion ; in other words the hypersensitivity of quantum dynamics under small perturbations, as measured by the type of decrease of the Quantum Fidelity, could be a signature of what is often called (rather improperly) "Quantum Chaos".

Thus it is highly desirable to have a better understanding of how this function of time (represented by equation (2) for suitable class of reference quantum states ψ) behaves at infinity in general as well in particular systems. In this paper we have been able to describe the full time-behavior of the Quantum Fidelity for a rather specific class of systems, and for reference states in a suitable class of "generalized coherent states. The underlying $SU(1,1)$ structure possessed by these systems allows us to perform an exact calculus of the Quantum Fidelity, and to compare it with the "Classical Fidelity" of the corresponding classical motion. This classical motion can be either stable, or unstable with positive Lyapunov exponent. The Quantum Fidelity has the following remarkable properties :

-either it decays to a (generally non-zero) limit in the past and in the future
-or it manifests an infinite sequence of recurrences to 1 as time evolves.

This sheds a new light on this question which has been addressed in a great variety of cases in the physics literature, and where the Quantum Fidelity is generally claimed to decay very rapidly to zero. Thus the first mathematical study presented here on the long time behavior of the Quantum Fidelity could allow in the future a better understanding of these features in more general situations on a rigorous level.

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